Seemingly Unrelated Regression (SUR) Models as a Generalized Least Squares (GLS) Solution to Path Analytic Models

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Multivariate Regression

$$\mathbf{Y}_{(N\text{x}p)} = \mathbf{X}_{(N\text{x}k)}\,\mathbf{B}_{(k\text{x}p)} + \boldsymbol{\varepsilon}_{(N\text{x}p)}$$

where $\mathbf{Y}$ is a matrix of p dependent variables, $\mathbf{X}$ is a k-dimensional design matrix, and $\boldsymbol{\varepsilon}$ is an error matrix, which is assumed to be distributed as:

$$\boldsymbol{\varepsilon}_{(N\text{x}p)} \sim N_{(N\text{x}p)}(\mathbf{0}, \Sigma \otimes \mathbf{I}N)$$

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Multivariate Regression

Multivariate regression theory using ordinary least squares (OLS) assumes that all of the **B** coefficients in the model are unknown and to be estimated from the data as:

$$\hat{\mathrm{B}} = (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{Y})$$

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SUR Models

Zellner (1962) formulated the Seemingly Unrelated Regression (SUR) model as *p* correlated regression equations, which has also been referred to as multiple-design multivariate (MDM) models (Srivastava, 1967).

The *p* regression equations are "seemingly unrelated" because taken separately the error terms would follow standard linear OLS linear model form.

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Multiple Univariate Regression

Calculating *p* separate standard OLS solutions ignores any correlation among the errors across equations; however, because the dependent variables are correlated and the design matrices may contain some of the same variables there may be “contemporaneous” correlation among the errors across the *p* equations.

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SUR Models

SUR models are applied when there are several equations, which appear to be unrelated but are related by the fact that:

(1) some coefficients are the same or assumed to be zero;

(2) the disturbances are correlated across equations; and/or

(3) a subset of right hand side variables are the same.

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SUR Models

SUR Models allow each of the *p* dependent variables to have a **different design matrix** with some of the predictor variables being the same.

SUR models allow for a variable to be **both** in the **Y** and **X** matrices, which has particular relevance to **Path Analysis**.

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SUR Model

$$E[\mathbf{Y}_{(N\text{x}p)}] = \{\mathbf{X1}_{(N\text{x}m1)}\,\beta_{\mathbf{1}\ (m1\text{x}1)}\ ,\ \mathbf{X_2}_{(N\text{x}m2)}\,\beta_{\mathbf{2}\ (m2\text{x}1)}\ ,\ \mathbf{X}_{\boldsymbol{j}(N\text{x}mj)}\,\beta_{\boldsymbol{j}(mj\text{x}1)}\ ,\ \mathbf{X}_{\boldsymbol{p}(N\text{x}mp)}\,\beta_{\boldsymbol{p}(mp\text{x}1)}\}$$

$$\underset{(Np\text{x}1)}{E(\mathbf{y}_v)} = \overset{E(\mathbf{y}_v)}{\underset{(Np\text{x}1)}{\begin{bmatrix} \hat{\mathbf{y}}_1 \\ \hat{\mathbf{y}}_2 \\ \ldots \\ \hat{\mathbf{y}}_j \\ \ldots \\ \hat{\mathbf{y}}_p \end{bmatrix}}} \begin{matrix} (N\text{x}1) \\ (N\text{x}1) \\ \\ (N\text{x}1) \\ \\ (N\text{x}1) \end{matrix} = \overset{\mathbf{D}}{\underset{(Np\text{x}M)}{\begin{bmatrix} \underset{(N\text{x}m_1)}{\mathbf{X}_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & \underset{(N\text{x}m_2)}{\mathbf{X}_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & & & \underset{(N\text{x}m_j)}{\mathbf{X}_j} & \ldots & \mathbf{0} \\ & (sym) & & & & \underset{(N\text{x}m_p)}{\mathbf{X}_p} \end{bmatrix}}} \overset{\mathrm{B}}{\underset{(M\text{x}1)}{\begin{bmatrix} \beta_1(m_1\text{x}1) \\ \beta_2(m_2\text{x}1) \\ \\ \beta_j(m_j\text{x}1) \\ \\ \beta_p(m_p\text{x}1) \end{bmatrix}}}$$

where $\boldsymbol{M}$ is the total number of parameters estimated over the p models,

$$\boldsymbol{M} = \sum_{j=1}^{p} m_j$$

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SUR Model

$$\underset{(M\text{x}1)}{\hat{\mathrm{B}}} = \underset{(M\text{x}Np)}{[\,\mathbf{D}'} \;\; \underset{(Np\text{x}Np)}{\mathbf{Q}^{-1}} \;\; \underset{(Np\text{x}M)}{\mathbf{D}]^{-1}} \;\; \underset{(M\text{x}Np)}{[\,\mathbf{D}'} \;\; \underset{(Np\text{x}Np)}{\mathbf{Q}^{-1}} \;\; \underset{(Np\text{x}1)}{\mathbf{y}_{\mathbf{v}}]}$$

Q is weight matrix based on the residual covariance matrix of the **Y** variables and is formed as:

$$\underset{(Np\text{x}Np)}{\mathbf{Q}} = \underset{(p\text{x}p)}{\hat{\Sigma}} \otimes \mathbf{I}_N$$

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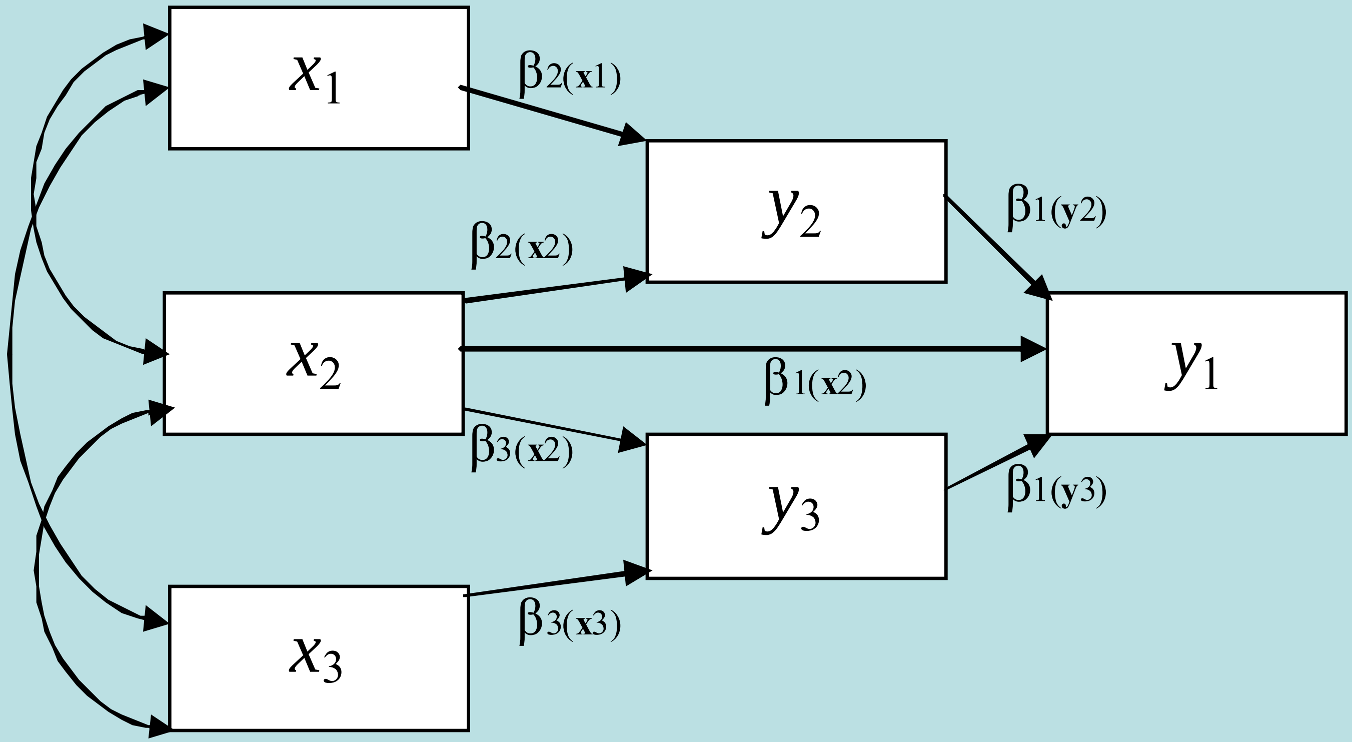
Path Model



$$
\begin{array}{lllllllll}
\hat{\mathbf{y}}_1 & = & \beta_{1(\mathbf{y}2)}\,\mathbf{y}_2 + & \beta_{1(\mathbf{y}3)}\,\mathbf{y}_3 + & \mathbf{0} & \mathbf{X}_1 & +\,\beta_{1(\mathbf{X}2)} & \mathbf{X}_2\,+ & \mathbf{0} \quad \mathbf{X}_3 \\
\hat{\mathbf{y}}_2 & = & & & \beta_{2(\mathbf{X}1)} & \mathbf{X}_1 & +\,\beta_{2(\mathbf{X}2)} & \mathbf{X}_2\,+ & \mathbf{0} \quad \mathbf{X}_3 \\
\hat{\mathbf{y}}_3 & = & & & \mathbf{0} & \mathbf{X}_1 & +\,\beta_{3(\mathbf{X}2)} & \mathbf{X}_2\,+ & \beta_{3(\mathbf{X}3)} \quad \mathbf{X}_3
\end{array}
$$

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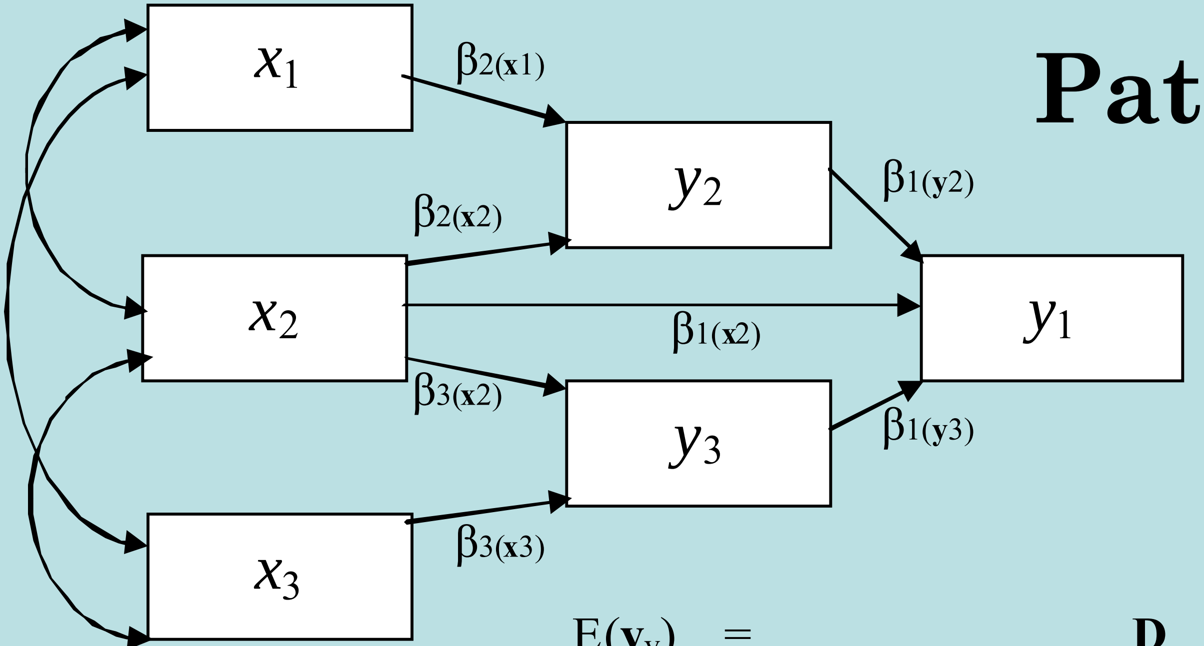
Path Model



$$
\underset{(3N\text{x}1)}{E(\mathbf{y}_v)} = \underset{(3N\text{x}1)}{\begin{bmatrix} \hat{\mathbf{y}}_{11} \\ \hat{\mathbf{y}}_{12} \\ \cdots \\ \hat{\mathbf{y}}_{1N} \\ \hat{\mathbf{y}}_{21} \\ \hat{\mathbf{y}}_{22} \\ \cdots \\ \hat{\mathbf{y}}_{2N} \\ \hat{\mathbf{y}}_{31} \\ \hat{\mathbf{y}}_{32} \\ \cdots \\ \hat{\mathbf{y}}_{3N} \end{bmatrix}}
= \underset{(3N\text{x}7)}{\overset{\mathbf{D}}{\begin{bmatrix}
\mathbf{y}_{21} & \mathbf{y}_{21} & \mathbf{x}_{21} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{y}_{22} & \mathbf{y}_{22} & \mathbf{x}_{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\cdots & \cdots & \cdots & & & & \\
\mathbf{y}_{2N} & \mathbf{y}_{21} & \mathbf{x}_{21} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
 & (N\text{x}3) & & \mathbf{x}_{11} & \mathbf{x}_{21} & \mathbf{0} & \mathbf{0} \\
 & & & \mathbf{x}_{12} & \mathbf{x}_{22} & \mathbf{0} & \mathbf{0} \\
 & & & \cdots & \cdots & & \\
 & & & \mathbf{x}_{1N} & \mathbf{x}_{2N} & \mathbf{0} & \mathbf{0} \\
 & (sym) & & (N\text{x}2) & & \mathbf{x}_{21} & \mathbf{x}_{31} \\
 & & & & & \mathbf{x}_{22} & \mathbf{x}_{32} \\
 & & & & & \cdots & \cdots \\
 & & & & (N\text{x}2) & \mathbf{x}_{2N} & \mathbf{x}_{3N}
\end{bmatrix}}}
\underset{(7\text{x}1)}{\overset{\mathbf{B}}{\begin{bmatrix} \beta_{1(\mathbf{y}2)} \\ \beta_{1(\mathbf{y}3)} \\ \beta_{1(\mathbf{x}2)} \\ \beta_{2(\mathbf{x}1)} \\ \beta_{2(\mathbf{x}2)} \\ \beta_{3(\mathbf{x}2)} \\ \beta_{3(\mathbf{x}3)} \end{bmatrix}}}
$$

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